1983

The variances of stochastic forecasts: calculating the reliability of predicted policy consequences

Suzanna Lynn Morris
Iowa State University

The variances of stochastic forecasts--calculating the reliability of predicted policy consequences

by

Suzanna Lynn Morris

A Thesis Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

MASTER OF SCIENCE

Department: Economics
Major: Agricultural Economics

Iowa State University
Ames, Iowa

1983

TABLE OF CONTENTS

CHAPTER I. INTRODUCTION

Objective

This thesis derives an exact formula for the variance of a forecast of an endogenous variable which is made with a forecast of an exogenous variable, and also presents formulas for variances of various measures of consumer welfare. The objective of this thesis is to provide these formulas as a means by which forecast researchers are able to report a predicted event along with a measure of the exact reliability of that prediction. Policy makers can then use this measure to their discretion when evaluating the accuracy of the predicted event.

Econometric Forecasting

Prior to the advent of the computer age, forecasting was regarded as a relatively simple science. In its earlier form, forecasting involved the use of human judgements by those who were learned in the field. The success of this process hinged on the individual expertise of the forecaster who was responsible for considering all relevant information and examining all pertinent indicators. A forecaster's effectiveness depended upon a sound intuitive feel for the commodity being forecasted. Specialization of study in each commodity was therefore necessary in order to forecast with a high level of accuracy.

With the application of econometrics, a more refined quantitative forecasting model evolved. Thus came estimations of more sophisticated models which had theoretical support. Econometrics was then the bridge between theory and real life. Economists could develop their skills to empirically test all relevant data, and not just those items which supported someone's preconceived notions. Econometrics made it easier to answer the "what will happen if" question quantitatively. A quote by Professor James Tobin reinforces the use of mathematics and statistics in forecasting:

> As an economist...I know that, bad as we are, we are better than anything else in heaven and earth at forecasting aggregate trends.... This statement is based on empirical experience.... (There is) a vindication of the hypothesis that there is no efficacious substitute for economic analysis in forecasting. Some maverick may hit a home run on occasion; but over the long season, batting averages tend to settle down to a sorry level when the esoteric methods of soothsaying are relied upon. (Cited in Ramsey, 1977:61)

The role of econometrics has been to apply statistical methods to measure economic relations. Model proliferation has resulted, thus allowing forecasting to evolve into a rigorous science. Forecast models have grown, becoming more complex and somewhat more artistic. A logical concern is then if these statistical models do indeed give statistically efficient forecasts.

Forecast users tend to put a lot of fairth in models which were carefully estimated by skilled econometricians. Predictive accuracy is then attributed to improvements in the basic model underlying the forecasts and to the experienced use of the econometric techniques. If every pertinent aspect has been considered and appropriately incorporated into the forecast model, then this should render a believable forecast.

A believable forecast is not always reliable though; and a forecast is not useful unless it is reliable.

Forecast Reliability Measures

What then constitutes a reliable forecast? A highly probable forecast could be considered accurate. Probabilities are often associated with a predicted event. Many times though, the probability is too vague to be used with any level of confidence. Statements such as "There's about a 60 percent probability that corn prices will increase $2 to $3 a bushel if the PIK program is enacted," and "There's about a 10 percent chance that consumer welfare will decrease by $2 billion if the program is legalized" are characteristic of many forecasts which are published in government reports. Statements such as these were written as conditional probabilities - that is, the probability that a given event will happen under a certain set of circumstances. They leave far too many unknowns unreported to be used in a policy setting context. Administrations are often constrained to use conclusions such as those written above because of the fact that adequate forecast evaluations are seldom done. We are left without really knowing exactly how dependable that forecast is.

Most of the forecast evaluation problem, which is also characteristic of econometric models in general, stems from the lack of a clear and accepted analytical basis for selecting the proper criteria on which to judge the models. With the government and businesses becoming more financially dependent upon forecasts, there is an increasing concern to find more precise and understandable methods of evaluation.

Dhrymes et al. (1972) has categorized the evaluation process on econometric models into two steps. The first relates the model to prior economic and statistical knowledge. This includes model construction and functional form. Ideally, this first step of model specification is directly correlated with the second step, model validation.

The purpose of validation is to verify that the model is fulfilling its stated purpose. In this context, validation is a model specific process. That is, the evaluation criteria for that model should be built into the method of estimation. If a tracking measure such as turning points is used to judge the accuracy of the model, then the validated model should minimize the number of points missed or falsely predicted. If all that is desired is a good fit of the model, then every estimated coefficient of the validated model should be significant (to a specified level). A validated model designed for forecasting should minimize the standard error of forecast.

Dhrymes et al. concedes that what model builders have done, to date, is to catalogue the predictive ability of models into different measures, concentrating on those aspects of the system which seem useful to them. These measures, i.e. tracking measures and standard errors, may or may not then be relevant to the model user's decision making. For example, the model estimated to accurately trace every turning point in the historical data, may not yield the best forecast of a continuing increase of a price. Furthermore, t statistics are more important in a model designed to test a specific hypothesis or measure some elasticity than in a model to be used for forecasting.

There are various other single variable measures which economists use to back up the goodness of the model. These include R-square, the coefficient of determination, mean square error (MSE) and its square root, the root mean square error (RMSE). A percent RMSE can also be calculated. Another widely used statistic, proposed by Theil, is called the coefficient of inequality.

Dhrymes et al. wrote that the predictive ability of a model is essentially a goodness-of-fit problem. It is from this belief that forecast model builders have found themselves restricted to the summary measures such as those in the preceding paragraph. These statistics only evaluate how well the model duplicates reality in a sample. A single equation econometric model may have a high R-square and very significant t statistics on all of the coefficients and still forecast very badly. The market may have undergone a structural change during the forecast period which the model was unable to predict. A large standard error of that forecast would have been enough indication of that change in the economy. It would imply then that the parameter estimates should be re-estimated before the forecast is published.

A problem with using measures such as RMSE and Theil's U-statistic for model evaluation is that they are not easily understood by all forecast users. It can be easily explained that the closer the Theil measure is to one, the worse the predictive performance of the model. This inherently and possibly incorrectly implies that every forecast from a model with a Theil-U measure of over .5 should be rejected. Another source of confusion may arise from understanding that the RMSE statistic measures the performance of a single model, while the percent RMSE is necessary to compare forecasts from different models. The point being that economists are often negligent in adequately interpreting to the decision makers exactly what the evaluation measures mean. Forecast users cannot then correctly apply confidence levels to forecasts.

A carry-over consideration is that economists themselves do not report the appropriate measures to evaluate how trustworthy an ex-ante[1] forecast is. A model with a low RMSE falsely suggests that an accurate forecast can be made with it. RMSE's are only a measure of simulation fit of the data in the sample set. Similarly, Theil's inequality coefficient can also only be applied to historical data or ex-post forecasts. It is this fact which spurred E.W. Streissler to write:

> A warning to those politicians and businessmen who have come to regard economic models...as some kind of panacea.... The aspect of all this that is most disconcerting to businessmen is that econometric forecasting has done positive harm by encouraging expectations for predictions that had little scientific justification. (Cited in Ramsey, 1977:103)

If the politicians and businessmen had the appropriate criteria with which to judge the exact reliability ex-ante forecast themselves, then they would not have to condemn the economists who make the inaccurate predictions. A simple number such as a standard error of a forecast would provide information that decision makers could use to decide for themselves whether to trust the forecast or not. Besides, since the

---

[1]An ex-ante forecast predicts values of the dependent variable beyond the estimation period. Alternatively, an ex-post forecast can be used to evaluate a forecast model by checking it against existing data.

standard errors of the estimated coefficients are so readily reported, the standard errors of the forecast should similarly be reported.

A typical econometric study done by Chambers and Just (1981) used five pages to report coefficients, standard errors and RMSE's for their 16 structural equations. Their study forecasted the effects of exchange rate changes on U.S. agriculture. After justifying the good fit of the models, their conclusion was

> The impact of a 10 percent depreciation of the exchange rate is substantial. Corn exports rise by over 90 million bushels, wheat exports by about 34 million bushels, and soybean exports by about 8 million bushels.

Their use of the word "about" makes one wonder how accurate the estimates really are, no forecast reliability measures were reported. Can the forecasts be trusted? Will all exports go up by the predicted amount if the exchange rate is depreciated? The value of the estimates as a policy tool is only as good as their reliability.

Macroeconomic models used to make forecasts are usually conditional upon certain policy options or economic conditions. The Chambers and Just study presented predictions of what might happen in the future under a given policy action. A forecast study may also predict what would have happened in the past if something had been different. A forecast is simply a prediction of an event which has not occurred under a given set of conditions. Those conditions may be either known or unknown. For example, a forecast of a decrease in the unemployment rate to 7 percent may be conditional upon the money supply growing 8 percent per annum and a forecast that government expenditure will increase 6 percent. That particular forecast depends on two things--that the money supply growth rate will not change and the forecasted expenditure increase. The accuracy of the unemployment rate forecast also depends on two things--the reliability of the expenditure forecast and the unemployment rate model goodness of fit.

There is little prior research done which recognizes that the reliability of that expenditure forecast is an important factor in calculating the accuracy of the unemployment rate forecast. Accounting for the variability of the conditioning variables can significantly increase the variance of a forecast. Care must be taken to distinguish an unconditional forecast, where values for all the explanatory variables in the forecasting equation are known with certainty, from a conditional forecast, where the variables are not known with certainty.

Martin Feldstein (1971) published a paper which presents the appropriate formula for the standard error of a forecast when the exogenous variables in the forecast period are stochastic. Ladd (undated) used formulas presented by Bohrnstedt and Goldberger (1969) to derive that same stochastic forecast variance formula. His paper, entitled "Variances of Products of Forecasts," also derives variance formulas for alternative measures of consumer welfare. The following chapters fully cover the derivation and the use of the formulas presented in the aforementioned papers.

CHAPTER II: THE ORDINARY LEAST SQUARES MODEL, ESTIMATION AND FORECASTING

Model Assumptions

An ordinary least squares regression model can be specified as follows:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \ldots + \beta_k X_{kt} + u_t$$

Or, in matrix form

$$Y = X\beta + u \qquad (1)$$

where

Y = n x 1 column vector of the dependent variable observations;

X = n x k matrix of the independent variable observations (with rank of k);

β = k x 1 column vector of the unknown parameter coefficients; and

u = n x 1 column vector of the multivariate normal residuals.

Assumption 1) The error term is assumed to be normally distributed with a zero expected value and constant variance.

$$u \sim N(0,\sigma^2 I) \text{ where } E(u_t, u_{t-i}) = \sigma^2 \text{ for } i=0$$

$$= 0 \text{ for all } i \neq 0.$$

An estimate for the unknown parameter coefficients, the β's, is given by b; b is the best linear unbiased estimate of the true parameter. The calculated parameter estimates equal

$$b = (X'X)^{-1} X'Y. \qquad (2)$$

The unconditional forecast of a value of Y is given by:

$$y_f = X_f'b + e_f \tag{3}$$

where

y_f = the estimated forecast value of Y;

X_f = 1 x k column vector of known forecast period values;

b = 1 x k column vector of the estimated parameters; and

e_f = the forecast error term.

To estimate the exact variance of this forecast, all sources of error must be accounted for. Four forecast error sources will be considered.

Error may result from model misspecification. This can imply that either a linear model was incorrectly transformed, a variable was omitted, or another was mistakenly used. Since the researcher is presumably very familiar with the data, the resulting forecast model must be accepted as the most accurate representation of the true model. Therefore, this source of error is assumed minimal.

Error will also result from usual stochastic disturbances. Although there is no pattern to the residuals if the model is correctly specified, the random nature of the errors will still remain. However, as stated in assumption 1, the expected mean value of the error term is zero. The estimated mean squared error, from the forecast model,

$$\hat{\sigma}^2 = s^2 = e'e/n-k$$

is included in the forecast variance.

Two other sources of error which must be accounted for when calculating the forecast standard error concern the X matrix and the β's. When any one of the independent variables is not known, an estimate for it must be used. A stochastic X matrix will introduce error when a forecast is made. For the time being, a fixed X matrix will be assumed, thus, an unconditional forecast.

The classification of a forecast with a stochastic X matrix as conditional is a matter of syntax. An OLS model is always conditional upon the expected value of the error term and the fit of the estimated regression parameters. Regression theory states that the coefficient estimates are normally distributed random variables and will, on the average, equal the true underlying parameters.

The remainder of this thesis will therefore define conditional forecasts as those forecasts made when some (or all) elements of X_f are unknown. When any estimates for either the exogenous variables of the coefficients are used, the variance of those estimates must be accounted for when estimating the forecast variance. The variance or the estimated coefficients are considered in the following section. Estimates for the exogenous variables will be considered in a later section.

The Coefficient Estimates Dispersion Matrix

The variance-covariance matrix of the coefficient estimates from equation (1) is equal to

$$\Delta(\beta) = E\{(b-B)(b-B)'\}$$

$$= (X'X)^{-1}\sigma^2.$$

An estimate of σ^2 is given by s^2. So,

$$D(b) = (X'X)^{-1}s^2. \tag{4}$$

D(b) is then an estimate of $\Delta(b)$.

$$D(b) = \begin{bmatrix} V(b_0) & C(b_0,b_1) & C(b_0,b_2) & \cdots & C(b_0,b_k) \\ & V(b_1) & C(b_1,b_2) & \cdots & C(b_1,b_k) \\ & & V(b_2) & \cdots & C(b_2,b_k) \\ & & & \ddots & \vdots \\ \text{symm.} & & & & V(b_k) \end{bmatrix}$$

Unconditional Forecasts

An unconditional forecast is made from an OLS model when values for all of the explanatory variables in the equation are known with certainty for the forecast period. This occurs most frequently when either lagged variables or binary (0-1) variables are used to estimate the equation. A predictable change in an independent variable is also a known forecast value.

If a forecast for one month ahead is being made, a variable such as population, which changes slightly every month, could be considered known. This assumption should not be made for a forecast further into the future though. A forecast of a price effect from a change in policy could still be unconditional if the proposed change is known.

<u>The variance formula for unconditional forecasts</u>

Equation (3) is unconditional when X_f is a vector of known numbers. The error of the unconditional forecast, Y, is given by

$$e_f = y_f - Y_f$$

$$= Xb - Y_f - u_f$$

$$= Xb - XB - u_f$$

$$= -u + X(X'X)^{-1}X'u.$$

Using (1) and (3) and assuming no autocorrelated or homoscedastic error terms, an estimate for the variance of the forecast error is then given by

$$\hat{\sigma}_f^2 = E(e_f^2)$$

$$= \sigma^2(1 + X_f'(X'X)^{-1}X_f).$$

Using equation (4)

$$V(y_f) = \hat{\sigma}_f^2 = X_f' \; D(b) \; X_f + s^2 \tag{5}$$

where

X_f = 1 x k row vector of known forecast period values;

D(b) = k x k dispersion matrix of the estimated β parameters; and

s^2 = the estimated mean square error from the forecast model.

The standard error of this forecast, $s(y_f)$, is the square root of the variance, $V(y_f)$. The standard error (i.e. the standard deviation of the forecast error) is used to calculate a tolerance region for the forecast.

Since a point forecast is a random variable and not a parameter, a confidence statement about a region around that random variable is called a tolerance interval. A percent level of significance is associated with that tolerance interval.

To compute the interval, first note that the forecast, y_f, is a linear combination of the estimated coefficients (which are random variables). The normality of the error term of this unconditional forecast (see assumption 1) implies normality of the b's. This, in turn, implies that y_f follows a normal distribution so a standard interval computation can be done. $(y_f - Y_f)/s(y_f)$ has a t distribution with n-k degrees of freedom (n is the number of observations and k is the number of estimated parameters). A $100(1-\alpha)$ percent tolerance interval is then:

$$y_f \pm t_{\alpha/2,n-k} s(y_f)$$

or

$$X_f' b + t_{\alpha/2,n-k} s(y_f).$$

The calculated standard error of the forecast, which is used to compute the interval, is a simple measure of reliability of that forecast. A large standard error will increase the size of the interval, thus implying that the forecast is less trustworthy. It is therefore very important to calculate the forecast variance correctly. A later section of this chapter develops the seldom recognized additional considerations necessary to calculate the correct variance for a conditional forecast.

The covariance between two unconditional forecasts

Suppose two different equations are used to forecast the supply of oranges in the U.S. One model forecasts the quantity of oranges from California, and the other, the supply from Florida. The estimates are then added together to forecast the U.S. supply. It is possible to compute the covariance of the two different forecasts.

The two equations, estimated by OLS are

$$Y_1 = X_1B_1 + u_1$$

$$Y_2 = X_2B_2 + u_2$$

where X_1 is n x k_1 and X_2 is an n x k_2 matrix of the independent variables. The forecasts are then

$$y_{1f} = X'_{1f}b$$

$$y_{2f} = X'_{2f}b$$

where

X_{1f} and b_1 are both 1 x k_1 column vectors; and

X_{2f} and b_2 are both 1 x k_2 column vectors of forecast period values.

The k_1 x k_2 dispersion matrix of the estimated parameters equals

$$D(b_1,b_2) = E\{(b_1 - \beta_1)(b_2 - \beta_2)'\} \tag{6}$$

$$= (X_1'X_1)^{-1}X_1' \; D(u_1u_2) \; X_2(X_2'X_2)^{-1}.$$

If elements of u_1 and u_2 are uncorrelated, $D(u_1u_2) = C(u_1,u_2)$, a scalar, and

$$D(b_1,b_2) = (X_1'X_1)^{-1}X_1X_2(X_2'X_2)^{-1}C(u_1,u_2).$$

$$D(b_1,b_2) = \begin{bmatrix} C(b_{10},b_{20}) & C(b_{10},b_{21}) & \cdots & C(b_{10},b_{2k_2}) \\ C(b_{11},b_{20}) & C(b_{11},b_{21}) & \cdots & C(b_{11},b_{2k_2}) \\ \vdots & \vdots & & \vdots \\ C(b_{1k_1},b_{20}) & C(b_{1k_1},b_{21}) & \cdots & C(b_{1k_1},b_{2k_2}) \end{bmatrix}$$

The covariance between the two forecasts y_1 and y_2 is then

$$C(y_{1f},y_{2f}) = X_{1f}'D(b_1,b_2)X_{2f} + C(u_1,u_2). \tag{7}$$

Note that if u_1 and u_2 are independent, $D(b_1,b_2)$ is a null matrix and y_1 and y_2 are uncorrelated.

The Bohrnstedt and Goldberger Formulas

Some important problems that economists encounter require simultaneous treatment of several random variables. A classic case is computing predicted revenue by summing price times quantity. If both price and quantity are predictions, then they are jointly distributed random variables.

Suppose revenue forecasts have been made from predicted sales of two different farm commodities. The revenue forecasts are products of forecasts themselves. Bohrnstedt and Goldberger (1969) have developed formulas which can be used to calculate both the exact variance and covariance of products of random variables.

By definition, the variance of the product of two random variables x and y is

$$V(xy) = E\{xy - E(xy)\}^2.$$

To obtain the exact variance, let x and y have expectations E(x) and E(y), variances V(x) and V(y) and covariance C(x,y), and let $E^2(x)$ = the square of E(x). If x and y are jointly normally distributed, then the formula for the exact variance of their product is given by

$$V(xy) = E^2(x)V(y) + E^2(y)V(x) + 2E(x)E(y)C(x,y) + V(x)V(y) + C^2(x,y). \tag{8}$$

Equation (8) reduces to the formula for the exact variance of the product of two stochastically independent random variables:

$$V(xy) = E^2(x)V(y) + E^2(y)V(x) + V(x)V(y). \tag{9}$$

For the covariance of products, let x, y, u, and v be jointly distributed random variables. By definition, the covariance of the products xy and uv is

$$C(xy,uv) = E\{xy - E(xy)\}\{uv - E(uv)\}.$$

If the four variables follow a multivariate normal distribution, the formula for the exact covariance of their products is given by:

$$C(xy,uv) = E(x)E(u)C(y,v) + E(x)E(v)C(y,u) + E(y)E(u)C(x,v) + E(y)E(v)C(x,u) + C(x,u)C(y,v) + C(x,v)C(y,u). \tag{10}$$

Formulas (8), (9), and (10) apply immediately if the random variables are forecasts. To obtain consistent estimates of V(xy) and C(xy,uv) replace variance, covariances and the expected values in the formulas with their consistent estimates.

Stochastic forecasts

When estimates, or forecasts, are used for any independent variable in an ordinary least squares model, the extrapolated prediction made from that model is conditional upon the estimates used. For example, if an ex ante forecast of farm marketing was used as an explanatory variable in a price forecast, then the price forecast is conditional upon the supply estimate. Furthermore, the reliability of that price forecast is also conditional upon the reliability of the supply estimate. A large variance of a conditioning variable should not be overlooked when calculating the reliability of a conditional forecast.

A forecast will include additional error when one or more of the forecast period exogenous variables are stochastic. Application of the formulas presented by Bohrnstedt and Goldberger yield the appropriate formula for the variance and covariance of stochastic forecasts. The derived formulas account for the additional variability from the conditionals.

The variance formula for a stochastic forecast

An equation similar to equation (3) gives an estimate of the conditional forecasted value of Y:

$$y_f = x_f'b + e_f \qquad (11)$$

where

y_f = the estimated value of Y;

x_f = 1 x k column vector of forecast period values (at least one is stochastic);

b = 1 x k column vector of the estimated β parameters; and

e_f = the forecast error term.

The difference between equations (3) and (11) is that X_f is now estimated by a vector x_f. Some predicted values in x_f must be used. The forecaster now has the problem to estimate y_f when neither β nor X_f is known, and to calculate the reliability of that estimate.

Let x_f be an asymptotically normal and consistent estimate of the actual value of X_f. The forecast of x_f has dispersion of

$$\Delta(x_f) = E\{(x_f - X_f)(x_f - X_f)'\}. \qquad (12)$$

Sufficient means for deriving the variable estimates only require that x be unbiased. The forecaster may then estimate the covariance matrix so $D(x_f)$ is an estimate of $\Delta(x_f)$.

$$D(x_f) = \begin{bmatrix} V(x_0) & C(x_0,x_1) & C(x_0,x_2) & \cdots & C(x_0,x_k) \\ & V(x_1) & C(x_1,x_2) & \cdots & C(x_1,x_k) \\ & & V(x_2) & \cdots & C(x_2,x_k) \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ \text{symm.} & & & & V(x_k) \end{bmatrix}$$

The elements in $D(x_f)$ are not the variances and covariances of the X_i's in the sample period. $V(x_i)$ and $C(x_i,x_j)$ are calculated for each of the forecast values of the independent variables. Construction of $D(x_f)$ this way explicitly shows how the reliability of the conditional forecast depends upon how good the independent variable forecasts are.

Some of the elements in x_f may be known with certainty. The intercept term, for example, is defined as unity. This implies that $D(x_f)$ will have rank of less than k. This will not effect the use of the conditional variance formula derived.

When both the b's and the x's are estimated, two additional considerations are assumed. Assumption 2) The estimated regression parameters are independent of the regressors, i.e. $E\{(x_f-X_f)(b-\beta)'\}=0$. Assumption 3) The residual terms are distributed independently of the stochastic regressors, i.e. $E(u_f'x_f)=0$.

To derive the formula for the exact variance of a forecast, y_f, first rewrite equation (11) in scalar notation:

$$y_f = \sum_{i=0}^{k} x_{if} b_i + e_f.$$

The forecast error, e_f, has expected value of zero:

$$e_f = Y_f - y_f$$

$$E(e_f) = E(Y_f - y_f)$$

$$= E(u_f)$$

$$= 0.$$

A consistent estimate for the variance of the forecast is then:

$$V(y_f) = \hat{\sigma}^2_{y_f} = V(\sum_{i=0}^{k} x_{if}b_i) + V(e_f)$$

$$= \sum_{i=0}^{k} V(x_{if}, b_i) + 2\sum_{i} \sum_{i \neq j} C(x_{if}b_i, x_{jf}b_j) + s^2$$

Noting that both the b's and the x's are random variables (they are estimates) then the Bohrnstedt and Goldberger equations can be used to evaluate the following expressions.

$$V(xy) = E\{xb - E(xb)\}^2$$

$$C(x_ib_i, x_jb_j) = E\{x_ib_i - E(x_ib_i)\}\{x_jb_j - E(x_jb_j)\}.$$

Equations (8) and (10) and assumptions 1 and 2 are applied, and estimated values are inserted to yield a consistent estimate for the exact variance of a stochastic forecast.

$$\hat{\sigma}^2_{y_f} = V(y_f) = \sum_{i}\{b_i^2 V(x_{if}) + x_{if}^2 V(b_i) + V(b_i)V(x_{if})\} +$$

$$2\sum_{i} \sum_{i \neq j} \{b_i b_j C(x_{if}, x_{jf}) + x_{if}x_{jf}C(b_i, b_j) +$$

$$C(b_i, b_j)C(x_{if}, x_{jf})\} + s^2.$$

Using vector matrix notation, this equation can be written as

$$V(y_f) = x_f' D(b) x_f + s^2 + b' D(x_f) b + tr\{D(b)D(x_f)\} \qquad (13)$$

where

x_f = 1 x k column vector of forecast period numbers (at least one is stochastic);

$D(b)$ = k x k dispersion matrix of the estimated coefficient;

b = 1 x k column vector of the estimated β parameters;

$D(x_f)$ = k x k dispersion matrix of the stochastic exogenous variables;

tr(M) = the trace of matrix M = sum of terms on the main diagonal of M; and

s^2 = the estimated mean square error for the OLS model.

If values of x_f are all known numbers, $x_f = X_f$, then $D(x_f)$ will be a null matrix. Equation (13) will then reduce to the standard forecast variance formula, equation (5).

When only one element in the x_f vector is a forecast, then a simplified form of equation (13) can be used. This is because $D(x_f)$ will have only one element. If x_j is a forecast, $D(x_f)$ will be a k x k matrix with $V(x_j)$ in its j x j position and zero's elsewhere. The forecast variance is then

$$V(y_f) = x_f'D(b)x_f + s^2 + \{b_j^2 + V(b_j)\}V(x_j). \tag{14}$$

Once the variance of a stochastic forecast has been calculated, a tolerance interval for that forecast can be found. Recall that when X_f was a set of known numbers, then a t distribution was used. When x_f is stochastic, however, then y_f is not distributed normally and the calculated t cannot be used. The sum of the product of pairs of normal variables $(\sum_i x_{if} b_i)$ will instead follow a multivariate nonnormal distribution. Feldstein's proposal of a simpler alternative is to make no assumptions about the distribution of the error term and the x_{if}'s and to define an outer bound forecast interval by the Chebychev inequality:

$$P\{|y_f - Y_f| \geq k\ s(y_f)\} \leq 1/k^2$$

where k is any positive constant. This tolerance interval for the random variable y_f is interpreted differently from the classical interval definition. This inequality might be read: The probability that the observed value of Y_f will fall outside the interval $y_f \pm k\, s(y_f)$ does not exceed $1/k^2$. (It should be noted that the use of this generalized definition for a tolerance interval will yield a wide interval relative to the magnitude of the forecast. This will occur especially when the estimated error is large.)

The equation for the variance of a stochastic forecast (equation (13)) can be used to show that as the forecast target date moves farther into the future, the reliability of that forecast will deteriorate. To see this, assume a simple OLS model

$$Y_t = \beta_0 Y_{t-1} + \beta_1 X_t + u_t$$

where assumption 1 applies.

If i = 1, 2, ... T (the sample set), forecasts are to be made for periods T+i, or y_{T+i}. Assume all X_{T+i} are known with certainty, that is assume $X_{T+i} = X_T$ for all i. Furthermore, assume $y_{T+i} = Y_T$ for all i.

The forecast for period T + 1 is

$$y_{T+1} = b_0 Y_t + b_1 X_{T+1}.$$

The variance of the forecast, using equation (5), is

$$V(y_{T+1}) = Y_t^2 V(b_0) + X_{T+1}^2 V(b_1) + 2Y_t X_{T+1} C(b_0, b_1) + s^2. \qquad (15)$$

The forecast for time period T + 2 is

$$y_{T+2} = b_0 y_{T+1} + b_1 X_{T+2}.$$

Since only one independent variable is estimated, equation (14) can be used to calculate the variance.

$$V(y_{T+2}) = V(y_{T+1}) + b_0^2 V(y_{T+1}) + V(b_0)V(y_{T+1})$$

$$= V(y_{T+1})\{1 + b_0^2 + V(b_0)\}. \qquad (16)$$

The forecast for time period T + 3 is

$$y_{T+3} = b_0 y_{T+2} + b\ X_{T+3}.$$

Equation (14) can be used again to calculate the variance of this forecast. $D(x_f)$ will be a 2 x 2 matrix with $V(y_{T+2})$ in the first row and column. Notice that $V(y_{T+1})$ is included in $V(y_{T+2})$. So the simplified variance of the forecast for time period T + 3 is

$$V(y_{T+3}) = V(y_{T+1})\{1 + b_0^2 + V(b_0) + \{(b_0^2 + V(b_0)\}^2\}. \qquad (17)$$

The variance of a T + i forecast will always increase as i approaches infinity. A general formula can be written which illustrates this.

$$V(\hat{Y}_{T+j}) = \sum_{j=0}^{J-1} V(\hat{Y}_{T+1})\{b^2 + V(b)\}^j$$

$$= V(\hat{Y}_{T+1}) + \{b^2 + V(b)\}V(\hat{Y}_{T+J-1}). \qquad (18)$$

The covariance between two stochastic forecasts

Suppose two forecast models are estimated using OLS.

$$Y_1 = X_1 B_1 + u_1$$

$$Y_2 = X_2B_2 + u_2$$

Conditional predictions are then obtained from the models:

$$y_{1f} = x'_{1f}b_1 + e_{1f}$$

$$y_{2f} = x'_{2f}b_2 + e_{2f}$$

where

x_{1f} and b_1 are both 1 x k_1 column vectors;

x_{2f} and b_2 are both 1 x k_2 column vectors of estimated forecast period values; and

$E(e_{1f}) = 0$ and $E(e_{2f}) = 0$.

The covariance between the two forecasts is given by

$$C(y_{1f},y_{2f}) = E\{(y_{1f} - Y_{1f})(y_{2f} - Y_{2f})'\}.$$

The same argument used to derive equation (13) can be used to calculate the exact covariance.

$$C(y_{1f},y_{2f}) = C(\sum_i x_{if}b_{1i} - u_{1f},\sum_j x_{jf}b_{2j} - u_{2f})$$

$$= \sum_{ij}\sum C(x_{if}b_{1i},x_{jf}b_{2j}) + C(u_1,u_2).$$

Define

$$D(b_1,b_2) = E\{(b_1-\beta_1)(b_2-\beta_2)'\}$$

$$D(x_{1f},x_{2f}) = E\{(x_{1f}-X_{1f})(x_{2f}-X_{2f})'\}.$$

So,

$$C(y_{1f},y_{2f}) = x'_{1f}D(b_1,b_2)x_{2f} + b'_1D(x_{1f},x_{2f})b'_2 + \mathrm{tr}\{D(b_1,b_2)D(x_{1f},x_{2f})\} + C(u_1,u_2). \quad (19)$$

It is also possible to derive expressions similar to equations (7) and (19) for the covariance between a change in a variable, dy_{1f}, and another variable, y_{2f}.

$$C(dy_{1f},y_{2f}) = dx_{1f}D(b_1,b_2)dx_{2f} \quad (20)$$

or,

$$C(dy_{1f},y_{2f}) = dx_{1f}D(b_1,b_2)x_{2f} + b_1D(dx_{1f},dx_{2f})b_2 + \mathrm{tr}\{D(b_1,b_2)D(dx_{1f},dx_{2f})\}. \quad (21)$$

Variations of the Bohrnstedt and Goldberger formulas

As demonstrated in the previous section, the variance and covariance formulas presented by Bohrnstedt and Goldberger (1969) can have a number of applications. The formulas for the products of random variables become even more useful when they are specialized in a variety of ways. Assume a normal distribution of all random variables used. If x=u and y=v, then

$$C(xy,xy) = V(xy).$$

If x=y, then

$$V(x^2) = 2V^2(x) + 4E^2(x)V(x) \quad (22)$$

$$C(x^2,uv) = 2\{E(x)E(u)C(x,v) + E(x)E(v)C(x,u) + C(x,u)C(x,v)\}. \quad (23)$$

If x=y and u=v, then

$$C(x^2,u^2) = 4E(x)E(u)C(x,u) + 2C^2(x,u)$$

$$= 2\{2E(x)E(u) + C(x,u)\}C(x,u). \quad (24)$$

IF u=1, then

$$C(xy,v) = E(x)C(y,v) + E(y)C(x,v). \quad (25)$$

If u=1 and x=v, then

$$C(xy,x) + E(x)C(y,x) + E(y)V(x). \quad (26)$$

C(xy,xv) comes from setting x=u, and C(xy,uy) comes from setting y=v. If y=st, then V(xst) and C(xst,uv) can be derived by applying equation (8) twice if the product of st is normally distributed.

Variations of Variances

Regression theory

A simple estimated ordinary least squares regression model is given by

$$Y = b_0 + b_1X_1 + b_2X_2$$

where the b's are the best linear unbiased parameter estimates.

The covariance between the dependent variable, Y, and an independent variable, X_2, is equal to

$$C(Y,X_2) = E(\{X_2-E(X_2)\}\{b_0+b_1X_1+b_2X_2 - E(b_0+b_1X_1+b_2X_2)\})^2.$$

If $\overline{X}_i$ is an unbiased estimator for X_i, then

$$C(Y,X_2) = E(\{X_2-\overline{X}_2\}\{b_1(X_1-\overline{X}_1) + b_2(X_2-\overline{X}_2)\})$$

$$= b_1C(X_1,X_2) + b_2V(X_2). \quad (27)$$

The covariance between the dependent variable, Y, and an estimated coefficient, b_2, is equal to

$$C(Y,b_2) = C(b_0,b_2) + C(b_1X_1,b_2) + C(b_2X_2,b_2). \quad (28)$$

where

$$C(b_2X_2,b_2) = X_2V(b_2) \quad (29)$$

$$C(b_1X_1,b_2) = b_1C(X_1,b_2) + X_1C(b_1,b_2). \quad (30)$$

In using OLS, the assumption that the X's are independent of the b's is made. So, $C(X_1,b_2) = 0$ and

$$C(b_1X_1,b_2) = X_1C(b_1,b_2). \quad (31)$$

Substituting equations (29) and (31) into (28) yields

$$C(Y,b_2) = C(b_0,b_2) + X_1C(b_1,b_2) + X_2V(b_2). \quad (32)$$

To carry an example one step further, instead of one model, assume a simple two equation recursive model where forecasts for Y_1 and Y_2 are made by:

$$Y_{1f} = b_{10} + b_{11}X_{1f} + b_{12}X_{2f}$$

$$Y_{2f} = b_{20} + b_{21}X_{3f} + b_{22}Y_{1f}.$$

Assume X_{1f}, X_{2f}, and X_{3f} are known values. So the forecast of Y_{2f} is conditional only upon the Y_{1f} forecast. Say there was a change in exogenous variable X_{1f} such that it causes Y_{1f} to change. What then would be the covariance of that change in the Y_1 forecast, dY_{1f}, and Y_{2f}?

$$C(Y_{2f},dY_{1f}) = C(b_{20},b_{11}dX_{1f}) + C(b_{21}X_{2f},b_{11}dX_{1f}) + C(b_{22}Y_{1f},b_{11}dX_{1f}) \quad (33)$$

Use equation (25) to find

$$C(b_{20},b_{11}dX_{1f}) = b_{11}C(dX_{1f},b_{20}) + dX_{1f}C(b_{11},b_{20}) = 0. \qquad (34)$$

Equation (34) is equal to zero because the change in X_{1f}, dX_{1f}, is a constant, so its covariance is zero. Also, the covariance between coefficients in two different equations are zero. This is implied because in using OLS, errors in different equations are assumed to be distributed independently.

Using equation (10) and the assumptions directly above

$$C(b_{21}\ X_{21},b_{11}dX_{11}) = 0. \qquad (35)$$

The remaining variable of interest in equation (33) is

$$C(b_{22}Y_1,b_{11}dX_{11}) = b_{22}dX_{11}C(Y_1,b_{11}). \qquad (36)$$

Apply equation (32) to calculate $C(Y_1,b_{11})$ the substitute this derivation into (36) above. And finally, equation (33) becomes

$$C(Y_2,dY_1) = b_{22}dX_{11}\{C(b_{10},b_{11}) + X_{11}V(b_{11}) + X_{12}C(b_{11},b_{12})\}. \qquad (37)$$

The variance of the difference between two stochastic forecasts

Suppose that a forecast of \$62.00 per cwt. is made for the price of choice steers in Omaha. Furthermore, suppose there is a foreseeable sudden decrease in the supply of cattle shipped to Omaha such that an alternate forecast of \$65.00 per cwt. is made. It is possible to compute the variance of the \$3.00 difference between the two stochastic forecasts.

The two forecasts are made from the same OLS model:

$$Y_t = X\beta + u_t.$$

The forecasts are then

$$y_{1f} = x'_{1f}b$$

$$y_{2f} = x'_{2f}b \tag{38}$$

where x_{1f} and x_{2f} are both 1 x k column vectors with different elements of stochastic forecast period values.

The difference between the two forecasts is given by

$$dy_f = y_{1f} - y_{2f}.$$

The difference can be calculated by

$$dy_f = dx'_f b \tag{39}$$

where dx'_f denotes a vector of the differences in values of elements x_{1f} and x_{2f} ($dx_f = x_{1f} - x_{2f}$).

The variance of the difference is then

$$V(dy_f) = V(y_{1f}) + V(y_{2f}) - 2C(y_{1f},y_{2f})$$

$$= X'_{1f}D(b)X_{1f} + s^2 + X'_{2f}D(b)X_{2f} + s^2 - 2(X'_{1f}D(b)X_{2f} + s^2)$$

$$= dX'_f D(b)dX_f.$$

This same result is calculated by a different derivation

$$dy_f = y_{1f} - y_{2f}$$

$$= X'_{1f}b - X'_{2f}b$$

$$= (X'_{1f} - X'_{2f})b$$

$$= dX_f b$$

$$V(dy_f) = dX'_f D(b)dX_f. \tag{40}$$

The variance formula (40) is very similar to the unconditional variance formula (equation (5)). It can be shown that a similar argument used to derive the variance formula for stochastic forecasts (equation (13)) will yield

$$V(dy_f) = dx_f' D(b) dx_f + b' D(dx_f) b + tr\{D(b) D(dx_f)\}. \tag{41}$$

This formula accounts for the stochastic nature of the forecasts made with equation (38).

The variance of the ratio of two forecasts

A variable is usually divided by a deflator whenever real values are desired. The prices in a quantity dependent demand equation, for example, are divided by a price index to obtain constant dollar amounts. Disposable personal income is also usually divided by the implicit price deflator. If predictions were used for both the nominal value of income and the deflator, then statistical differentials can be used to calculate the variance of the real disposable income forecast.

Define w as the ratio of two joint normal variables, $w = x/y$. Application of first order statistical differentials (Rao, 1973) yields as the variance of x/y:

$$V(x/y) = V(x)/y^2 - 2xC(x,y)/y^3 + x^2 V(y)/y^4. \tag{42}$$

Variance Formulas for Some Measures of Consumer Welfare

Public policy decision makers are required to draw conclusions from the forecasts made by economists. They practice welfare economics--the optimum allocation of society's resources and what can be done to make this allocation more ideal.

A topic directly related to welfare economics deals with the actual measurement of the welfare cost due to market imperfections, or changes in the structure of the market. A rigorous calculation of welfare loss requires assumptions regarding utility functions and income redistribution. An article by Currie et al. (1971) provides the steps and caveats necessary to prove that the loss of consumer surplus equates to a loss of consumer welfare. Economists can then turn to measuring that area under the demand curve and above the price line called consumer surplus. But how accurate are these measurements?

A study done by Parker and Conner (1979) computes and compares annual consumer loss estimates for U.S. food manufacturing industries using three different and independent methodological approaches. The authors claimed strength to their conclusion since all three of the estimates displayed a significant convergence that consumer loss due to monopoly ranged from \$12 to \$14 billion. The first approach was a best-guess estimate, the other two estimates were derived from regression analysis. In assessing the goodness of one of the regressions, the authors write

> The estimated price difference regression equation supported the hypothesized predictions for each of the variables. All the terms were statistically significant, and the fitted equation explained nearly three-fourths the variation in the private label-national brand price differences. The good fit strongly supports our hypothesis that national brand price premiums are related to competitive conditions in food-manufacturing industries, and because of this we feel it provides a means for calculating reliable and disaggregated estimates of monopoly overcharges.

They go on to say that the estimates from the regressions are "believed to be more reliable" than the best-guess estimate, and they "should tend to reduce the error range of the overall loss estimate." Although "the authors have no method for estimating the likely error range of their (regression) methods, they feel that a 25 percent error on the two estimates would reasonably be expected." The major conclusion was "the large dollar loss amounts suggest a high payoff for increased policy attention to competitive problems in the food-manufacturing industry."

Studies such as the Parker and Conner (1979) article and others which predict consequences of policy outcomes support the argument that a more precise reliability measure is needed. A model with very significant coefficients which yields an estimate with a projected 25 percent error margin may not be accurate enough to use in a policy setting context.

Three measures commonly used in policy evaluations are studied in this section. The formulas presented by Bohrnstedt and Goldberger (1969) are then applied in order to calculate the exact variance of the consumer surplus measures. The standard error (the square root of the variance) can then be computed to be used as an accuracy index for users of the forecast.

The Laspeyre Variation

One of the more readily computable and widely used measures of consumer surplus is the Laspeyre Variation (LV). LV is the change in income required to purchase the original quantities of all goods after prices have changed.

$$LV = \sum_i q_i dp_i \tag{43}$$

where

q_i = quantity of product i consumed; and

dp_i = the change in price of product i.

If dp_i is an estimated change in price from a policy action and q_i is the resulting forecasted quantity demanded, then $L\hat{V}$ can be computed. The estimates have variances $V(q_i)$ and $V(dp_i)$ and covariance $V(q_i,dp_i)$. The variance of LV is

$$V(LV) = \sum_i V(q_i dp_i) + 2\sum_i \sum_{i\neq j} C(q_i dp_i, q_i dp_i).$$

Application of equations (5) and (13) yields an estimate for the exact variance.

$$\hat{V}(LV) = \sum_i \{q_i^2 V(dp_i) + dp_i^2 V(q_i) + 2q_i dp_i C(q_i,dp_i) + V(q_i)V(dp_i) + C^2(q_i,dp_i)\} + 2\sum_i \sum_{i \neq j} \{q_i q_j C(dp_i,dp_j) + q_i dp_j C(dp_i,q_j) + dp_i q_j C(q_i,dp_j) + dp_i dp_j C(q_i q_j) + C(q_i,q_j)C(dp_i,dp_j) + C(q_i,dp_j)C(dp_i,q_j)\} \quad (44)$$

If only one product is being considered, then LV = qdp and

$$V(LV) = q^2 V(dp) + dp^2 V(q) + 2qdpC(q,dp) + V(q)V(dp) + C^2(q,dp). \quad (45)$$

Consumer gain

Another simple measure of consumer surplus is Winch's Consumer Gain (CG). Winch (1965) interpreted Hick's methods measuring consumer's gain (or loss) from a fall (or rise) in a commodity price in terms of the amount of money which would offset the gain (or loss). The problem, he claimed, is that the amount of offset money is not the same thing as the gain or loss itself. In attempting to approximate a closer measurement of the gain (loss), Winch included what some authors refer to as deadweight loss.

$$CG = \sum_i \{q_i dp_i + (dq_i dp_i)/2\} \quad (46)$$

where

q_i = quantity of product i consumed;

dp_i = the change in price of product i; and

dq_i = the change in quantity of product i.

In addition to the allocational effect, $\Sigma q_i dp_i$, there is also a distributional effect, $\Sigma dq_i dp_i/2$, from the change in market structure.

If q, dq, and dp are all estimates, then $\hat{CG}$ for one product is

$$\hat{CG} = qdp + dqdp/2. \tag{47}$$

The variance of CG is then

$$V(CG) = V(qdp) + V(dqdp)/4 + C(qdp,dqdp). \tag{48}$$

This equation can be expanded to obtain an estimate for the exact variance. (Note that LV is a component of CG.)

$$\begin{aligned} V(CG) = V(LV) &+ \{dq^2V(dp) + dp^2V(dq) + 2dqdpC(dq,dp) + \\ &V(dq)V(dp) + C^2(dq,dp)\}/4 + qdqV(dP) + qdPC(dp,dq) + \\ &dPdqC(q,dp) + dp^2C(q,dq) + C(q,dq)V(dp) + \\ &C(q,dp)C(dp,dq). \end{aligned} \tag{49}$$

The area under the demand curve

The original and perhaps the most easy to understand measure of consumer surplus stems from Marshall's concept of the triangular area under the demand curve. Define consumer surplus as the area under the demand curve and above the price line. Write a price dependent demand equation

$$P = \beta_0 + \sum_{i=1}^{k-1} \beta_i X_i + \beta_k Q + u \tag{50}$$

where

P = the price of the product;

X_i = known exogenous variables which influence the price; and

Q = quantity of the product consumed.

If b_k is an estimate of β_k and Q is a forecast, $Q = q_f$, then estimated consumer surplus equals

$$\hat{CS} = -\frac{b_k q_f^2}{2} \quad (51)$$

If b_k and q_f are normally and independently distributed, then the variance of the estimated consumer surplus can be obtained using equations (8) and (10) and from $E(x^2) = V(x) + E^2(x)$.

$$V(CS) = \{b_k^2\{2V^2(q_f) + 4q_f^2V(q_f)\} + \{V(q_f)+q_f^2\}^2V(b_k) +$$

$$V(b_k)\{2V^2(q_f) + 4q_f^2V(q_f)\}\}/4 \quad (52)$$

The estimated variance of consumer surplus is obtained by replacing $V(q_f)$ and $V(b_k)$ with their estimates $\hat{V}(q_f)$ and $\hat{V}(b_k)$.

Note that if q_f is derived from a reduced form of a system of equations and b_k is a structural parameter estimate in that same system, then the restriction that b_k and q_f must be distributed independently is not met. This is because the reduced form coefficients used to derive q_f are functions of b_k. Statistical differentials must then be used to estimate V(CS). Applying equation (42) yields

$$V(\hat{C}S) = V(q_f^2)/4b_k^2 = q_f^2C(q_f^2,b_k)/2b_k^2 + q_f^4V(b_k)/4b_k^4. \quad (53)$$

Numerous applications of the Bohrnstedt and Goldberger formulas (equations (8) and (10)) were used in the derivation of the formulas for the variance of the estimated consumer surplus measures. Care must be taken to insure that the assumptions for the use of these formulas are met before the statistical procedures are used. Otherwise, some specification error will result. Proper application of the variance formulas will yield the correct value to use when judging the reliability of the surplus forecasts.

The multitude of forecast articles published seldom, if ever, present any exact reliability measures. Economists do, however, systematically report the well-behaved nature of the underlying models used to make the forecasts. Calculation of the standard error of a forecast may show forecasts to be statistically unreliable, and therefore of questionable value.

The following chapter is a good illustration of the concept reinforced throughout this thesis--the exact reliability of a forecast should be reported along with the forecast itself. The derived variance formulas are used to calculate the exact reliability of some hypothesized policy options in the wheat sector.

CHAPTER III. AN APPLICATION TO THE U.S. WHEAT SECTOR

Structural Equations

A publication written by Mo (1968) entitled "An Economic Analysis of the Dynamics of the United States Wheat Sector" models the demand structure of a major agricultural sector in the U.S. An objective of Mo's study was to forecast U.S. wheat utilization under different prospective government programs. This thesis will use the same estimation procedures and the same data that Mo used to make forecasts. It will then measure the reliability of those predicted consequences of public policies, something that Mo did not do.

The basic model is a simple dynamic recursive system. Five equations are used to make forecasts of exogenous variables. Those forecasts are then used, along with a specified support price, to predict six endogenous variables. Table 1 defines each of the variables used.

Annual data from 1928 through 1964 is used to estimate each of the following structural equations. β_{ij} denotes the j^{th} coefficient of an endogenous variable in equation i. An estimate for β_{ij} is given by b_{ij}. γ_{ij} denotes the j^{th} coefficient of an exogenous variable in equation i. An estimate of γ_{ij} is given by c_{ij}. The symbol u_{it} denotes the error in equation i at time t. Mo's structural equations can be written as

Mo (2.1) $$P_t = \gamma_{10} + \gamma_{11}P_{st} + \gamma_{12}K_tP_{ot} + u_{1t}$$

Mo (2.4) $$q_{ht} = \gamma_{20} + \beta_{21}P_t + \gamma_{22}P_{ct} + \gamma_{23}G(I_t) + u_{2t}$$

Table 1. Definitions of the variables used in the wheat sector model

Exogenous variables

$G(I_t)$	- A nonlinear transformation of variable I.
I_t	- Per capita disposable income at time t (dol. per capita).
L_t	- Grain consuming animal units of livestock fed annually at time t (mil. units).
O_t	- Total U.S. wheat production at time t (mil. bu.).
P_{ct}	- Consumer price index at time t (1957-59 = 100).
P_{ot}	- Farm price index of other feed grains (corn, oats, barley, and sorghum) at time t (1957-59 = 100).
P_{st}	- Average wheat support price at time t (dol. per bu.).
$D_t=1$	during WW II
$=0$	otherwise.
$K_t=1$	if there is no price support program at time t
$=0$	otherwise.
$\overline{K}_t=1$	if there is a government price support program at time t
$=0$	otherwise.

Endogenous variables

C_{ct}	- Commercial wheat inventory at the end of time t (mil. bu.).
C_{gt}	- Government wheat inventory at the end of time t (mil. bu.).
P_t	- Average wheat price received by farmers at time t (dol. per bu.).

Table 1. (Continued)

q_{Et}	-	Total U.S. export of wheat at time t (mil. bu.).
q_{ft}	-	Domestic use of wheat for feed at time t (mil. bu.).
q_{ht}	-	Domestic per capita use of wheat for food at time t (bu. per capita).

Mo (2.8) $q_{ft} = \gamma_{30} + \gamma_{31}P_{ot} + \gamma_{32}L_t + \gamma_{33}D_t + \beta_{34}P_t + u_{3t}$

Mo (2.12) $C_{gt} = \gamma_{40} + \gamma_{41}P_{st} + \gamma_{42}\overline{K}_t D_{t-2} O_t + \gamma_{43}C_{gt1} + u_{4t}$

Mo (2.16) $C_{ct} = \gamma_{50} + \gamma_{51\ ct-1} + \beta_{52}P_t + \beta_{53}C_{gt} + u_{st}$

Mo (2.20) $q_{Et} = \gamma_{60} + \gamma_{61}(C_{ct-1}+C_{gt-1}) + \gamma_{62}q_{Et-1} + \beta_{63}q_{ht} + u_{6t}$

The t subscripts date the variables. For example, let the 1964 value of P_{ot} equal P_{o4}. A 1965 forecast is $\hat{P}_{o5}$ and the actual 1965 value is given by P_{o5}. For ease of reading the equations in this chapter, any symbol without a time subscript will denote the forecasted 1965 value of the variable. Thus, P_o = 1965 value of P_{ot}, $\hat{P}_o$ = 1965 forecast of P_{ot}. The 1965 values of K, $\overline{K}$ and D equal their 1964 values. The 1965 values of C_{gt-1}, C_{ct-1}, and q_{Et-1} are equal to their 1964 values, C_{g4}, C_{c4}, and q_{E4}.

All of the estimated equations are given in Table 2. They are written out as they are traditionally presented with the standard errors in parentheses below the estimated parameters. The estimated mean square errors of the equations are also presented because they are used in calculating the variance of the forecasts. Ordinary least squares regression was applied to estimate the structural equations which predict the endogenous variables (equations 6-11 in Table 2). The equations for the exogenous variables (1-5) contained lagged exogenous variables and a time trend.

Note that three transformed variables are used in the structural equations: $K_t P_{ot}$, $\overline{K}_t D_{t-2} O_t$, and $G(I_t)$. Two involve multiplication of binary variables and the other, $G(I_t)$, is a nonlinear function of

Table 2. The estimated models used in the wheat sector study[a]

Equations for predicting exogenous variables

1. Farm price index

$$P_{ot} = 2.3006 + \underset{(0.1176)}{0.7426}\, P_{ot-1} + \underset{(0.4125)}{0.4899}\, T_t$$
$$R^2 = .6928 \qquad s^2 = 443.4214$$

2. Consumer price index

$$P_{ct} = 5.6138 + \underset{(0.0517)}{0.8464}\, P_{ct-1} + \underset{(0.1076)}{0.3908}\, T_t$$
$$R^2 = .9882 \qquad s^2 = 6.3477$$

3. Grain-fed livestock

$$L_t = 36.8670 + \underset{(0.1195)}{0.7275}\, L_{t-1} + \underset{(0.1456)}{0.1562}\, T_t$$
$$R^2 = .6359 \qquad s^2 = 63.5277$$

4. U.S. wheat production

$$O_t = 131.7288 + \underset{(0.1526)}{0.4363}\, O_{t-1} + \underset{(3.2781)}{9.3841}\, T_t$$
$$R^2 = .6172 \qquad s^2 = 21{,}735.68$$

5. Disposable income

$$I_t = -321.2771 + \underset{(0.0541)}{0.8183}\, I_{t-1} + \underset{(2.9395)}{12.3122}\, T_t$$
$$R^2 = .9936 \qquad s^2 = 2418.74$$

Estimated structural equations

6. Farm price and support price relation

$$P_t = 0.1507 + \underset{(0.0456)}{0.9181}\, P_{st} + \underset{(0.0014)}{0.0106}\, K_t\, P_{ot}$$
$$R^2 = .9484 \qquad s^2 = 0.0190$$

---

[a]Coefficient standard errors are given in parentheses.

Table 2. (Continued)

7. Food consumption relation

$$q_{ht} = 1.6656 - \underset{(0.0631)}{0.1834}\, P_t + \underset{(0.0041)}{0.0040}\, P_{ct} + \underset{(0.2186)}{1.3857}\, G\,(I_t)$$

$R^2 = .9547$ $s^2 = 0.0107$

8. Feed consumption relation

$$q_{ft} = -134.5848 - \underset{(37.6204)}{148.9124}\, P_t + \underset{(0.5807)}{1.6874}\, P_{ot} + \underset{(0.8863)}{1.7895}\, L_t + \underset{(34.3651)}{156.4021}\, D_t$$

$R^2 = .7798$ $s^2 = 2462.709$

9. Government inventory relation

$$C_{gt} = -30.5046 + \underset{(40.8497)}{44.5040}\, P_{st} + \underset{(0.0731)}{0.1242}\, K_t\, D_{t-2}\, O_t + \underset{(0.0788)}{0.7870}\, C_{gt-1}$$

$R^2 = .9121$ $s^2 = 21{,}360.45$

10. Commercial inventory relation

$$C_{ct} = 201.5054 - \underset{(24.4764)}{63.9081}\, P_t - \underset{(0.0279)}{0.0446}\, C_g + \underset{(0.1534)}{0.3583}\, C_{ct-1}$$

$R^2 = .7018$ $s^2 = 3288.467$

11. Export relation

$$q_{Et} = 411.3583 - \underset{(82.1604)}{106.1793}\, q_{ht} + \underset{(0.0651)}{0.0907}\, (C_{ct-1} + C_{gt-1}) + \underset{(0.1358)}{0.6603}\, q_{Et-1}$$

$R^2 = .8640$ $s^2 = 8813.394$

disposable income. Mo (1968) made a nonlinear transformation of the original income data, then fit a linear relation with the transformed variable.

Disposable income, I_t, is first forecasted using equation 5 in Table 2. The forecast is then substituted into the following transformation function to obtain $G(I_t)$:

$$G(I_t) = 6e^{-0.001I_t} - 5.7468e^{-0.002I_t}. \tag{54}$$

Objectives of the Study

This chapter applies some procedures for measuring reliability of forecasts of consequences of public policies. The equations in Table 2 are used to make forecasts of the variables. The equations developed in Chapter II are then applied to calculate the reliability of those forecasts. This chapter is divided into three studies:

In Study I, the objectives are to

(A) predict the 1965 values of endogenous variables if the 1964 wheat price support programs had been continued into 1965 and predict the reliability of the forecasts, and

(B) predict 1965 consumer surplus under 1964 policy.

In Study II, the objectives are to

(A) predict differences in 1965 endogenous variables if the 1965 wheat price support were \$0.10 higher than in 1964 and measure the reliability of the predictions, and

(B) predict the effects of the higher price upon consumer welfare and measure the reliability of the predictions.

In Study III, the objective is to

(A) predict the 1966 values of all variables if the 1964 wheat price support program had been continued into 1966 and measure the reliability of the forecasts. This study will demonstrate the deterioration in quality of forecasts as the forecast target date moves farther into the future.

In part A of Studies I and II, the focus is on comparing two measures of variance of forecast: a) the conventional measure (equation (5)), which is derived under the assumption that values of predictor variables (i.e. exogenous variables) are known, and b) a measure derived under the assumption that the values of the predictor variables are themselves predictions. Equation (13) is used to calculate the variance of stochastic forecasts.

Study I

Forecasts of 1965 variables and variances

In this study, the 1964 wheat support price program is assumed to be continued into 1965 ($P_{s4} = P_{s5} = \$1.32$ per bu.). The estimated equations from Table 2 were used to make 1965 forecasts of all of the variables. A functional relation to forecast the exogenous variables is given by

$$\hat{X}_5 = f(X_4, T_5).$$

Since lagged variables and a time trend variable are used in these equations, the forecasts are unconditional.

A recursive procedure is used to calculate the 1965 forecasts of the endogenous variables. The appropriate values of the independent variables are substituted into equations 6-11 to calculate the forecasts. These forecasts of the endogenous variables are conditional since the X_f' row vector is stochastic (it contains at least one predicted value).

After 1965 predictions for each of the variables were made, then the exact variance of each forecast was estimated. Equation (5) was applied to the unconditional forecasts and equations (5) and (13) to the stochastic forecasts. Table 3 presents a cumulative listing of the forecasts and variances calculated for Study I. Note that variance equation (13) is only used for the conditional endogenous variable forecasts.

The example given in this section computes a forecast for both the 1965 wheat price, $\hat{P}$, and per capita consumption, $\hat{q}_h$. The variances of the two forecasts are then calculated.

To compute the variance of the conditioning variable forecasts (regression equations 1-5), equation (5) is applied. The same equation can be used to find the variance of the wheat price forecast. This is because the support price, P_s, is known. Also, K_5 is known to equal zero since there is a price support program (see variable definitions list), so $K_5\hat{P}_{a5} = 0$. This implies that the X_f vector is known such that the 1965 wheat price forecast is unconditional.

The information necessary to calculate the price forecast and the variance of that forecast is:

1965 support price $P_{s5} = 1.32$

1965 value of K $K_5 = 0$

1965 farm price index forecast $\hat{P}_{o5} = 109.15$

The estimated MSE from model 6 $s^2 = 0.0190$

The 3x3 dispersion matrix of the estimated coefficients in model 6:

$$D(b) = \begin{bmatrix} 0.0059 & -0.0033 & 9.39\text{x}10^{-5} \\ & 0.0021 & 5.23\text{x}10^{-5} \\ \text{symm.} & & 2.07\text{x}10^{-6} \end{bmatrix} \qquad (55)$$

The 1965 wheat price forecast is

$$\hat{P} = X_f' b$$

where

$$X_f' = \begin{bmatrix} 1 & 1.32 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 0.1507 & 0.9181 & 0.0106 \end{bmatrix}$$

$$\hat{P} = \$1.36.$$

Using equation (5), the variance of this forecast is then

$$V(\hat{P}) = X_f' D(b) X_f + s^2$$

$= 0.0198$ and the standard error $s(\hat{P}) = 0.1408$.

Equation (13) must be used to compute the variance of the conditional forecasts. When calculating the variance of domestic per capita wheat consumption forecast, $V(\hat{q}_h)$, it becomes necessary to obtain an estimate for the variance of the nonlinear disposable income prediction,

$V(G(\hat{I}_t))$. The use of statistical differentials applied to equation (54) yields

$$V(G(\hat{I}_t)) = \left[\frac{\partial G(I_t)}{\partial I_t}\right]^2 V(\hat{I}_t)$$
$$= \left[0.01149e^{-0.002\hat{I}_t} - 0.006e^{-0.001\hat{I}_t}\right]^2 V(\hat{I}_t). \qquad (56)$$

(Throughout the rest of the chapter, the symbol $\hat{G}$ will be used in place of $G(\hat{I})$. Thus $V(\hat{G}) = V(G(\hat{I}))$.)

The information necessary to calculate $\hat{q}_h$ and $V(\hat{q}_h)$ is

1965 wheat price forecast	$\hat{P}$	= 1.36
The variance of the price forecast	$V(\hat{P})$	= 0.0198
1965 consumer price index forecast	$\hat{P}_c$	= 111.29
The variance of the price index forecast	$V(\hat{P}_c)$	= 7.08
1965 disposable income forecast	$\hat{I}$	= 2334.86
The variance of the income forecast	$V(\hat{I})$	= 2777.00
Income variable transformation using (54)	$G(\hat{I})$	= 0.5271
The variance of the transformation using (56)	$V(\hat{G})$	= $6.34\text{x}10^{-4}$
The estimated MSE from model 7	s^2	= 0.0107

The 4x4 dispersion matrix of the estimated coefficients in model 7

The 4x4 dispersion matrix of the forecasted independent variables

$$D(x_f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0.0198 & 0 & 0 \\ & & 7.08 & 0 \\ \text{symm.} & & & 6.34\text{x}10^{-4} \end{bmatrix} \quad (57)$$

The 1965 wheat consumption forecast is

$$\hat{q}_h = x_f'b$$

where

$$x_f' = \begin{bmatrix} 1 & 1.36 & 111.29 & 0.5271 \end{bmatrix}$$

$$b = \begin{bmatrix} 1.6656 & -0.1834 & 0.0040 & 1.3857 \end{bmatrix}$$

$$\hat{q}_h = 2.52 \text{ bushels per capita.}$$

The exact variance of this forecast is given by

$$V(\hat{q}_h) = x_f'D(b)x_f + s^2 + b'D(x_f)b + \text{tr}\{D(b)D(x_f)\}.$$

The variance calculation can be broken down into its components:

a. $x_f'D(b)x_f + s^2 = 0.0139$

b. $b'D(x_f)b = 1.996\text{x}10^{-3}$

c. $\text{tr}\{D(b)D(x_f)\} = \underline{2.251\text{x}10^{-4}}$

$0.0162 = V(\hat{q}_h)$.

Line a is the equivalent of equation (5), the traditional variance calculation. Lines b and c, which add up to 0.0023, show the additional variability present when the forecast is conditional. The last column in Table 3 is provided as an index to compare the two variance calculations. The ratio for the consumption forecast variance is

Table 3. Study I: Forecasts and variances

Exogenous variables	1964 value	1965 forecast	Forecast variance Using equation (5)	Using equation (13)	Ratio of equation (13) to equation (5)
P_o	107.00	109.15	507.54 (22.53)[a]	-	-
P_c	108.10	111.29	7.08 (2.66)	-	-
L	167.66	169.00	71.07 (8.43)	-	-
O	1290.47	1334.65	24,210.17 (155.79)	-	-
I	2268.00	2334.86	2,777.00 (52.71)	-	-
Endogenous variables					
P	1.38	1.36	0.0198 (0.14)	-	-
q_h	2.67	2.59	0.0139 (0.12)	0.0162 (0.13)	1.165
q_f	70.02	149.109	2,905.40 (53.90)	5,272.94 (72.61)	1.815
C_g	705.50	749.23	23,565.12 (153.51)	24,069.40 (155.14)	1.021
C_c	113.41	115.63	3,544.20 (59.53)	3,703.77 (60.86)	1.045
q_E	728.00	691.59	10,432.68 (102.14)	10,725.42 (103.56)	1.028

[a]Standard errors are given in parentheses.

0.0162/0.0139 = 1.165. The variable whose variance increased the most, q_f, has a ratio of 1.815. The magnitude of the ratio indicates how much less the measured reliability is when the variability of the conditionals is correctly accounted for.

Consumer surplus and its variance

The second objective in Study I is to predict 1965 consumer surplus under the 1964 price support policy. According to equation (50), Mo (2.4) must be rewritten as

$$P_t = (q_{ht} - \gamma_{20} - \gamma_{22}P_{ct} - \gamma_{23}G(I_t) - u_{2t})/\beta_{21}. \tag{58}$$

Let $b_k = 1/b_{21}$. Using equation (51) with $\hat{q}_h = 2.59$, consumer surplus equals

$$\begin{aligned} CS &= -\hat{q}_h^2/2b_{21} \\ &= 18.2558 \end{aligned} \tag{59}$$

In this model, $\hat{q}_h$ and b_{21} are not independent, so neither are $\hat{q}_h$ and $1/b_{21}$. Variance formula (53) must then be used to estimate V(CS).

$$V(CS) = V(\hat{q}_h^2)/4b_{21}^2 - \hat{q}_h^2 C(\hat{q}_h^2, b_{21})/2b_{21}^3 + \hat{q}_h^4 V(b_{21})/4b_{21}^4 \tag{60}$$

Apply equation (22) to compute $V(\hat{q}_h^2)$. To obtain $C(q_h^2, b_{21})$, set x=y in equation (25).

$$\begin{aligned} C(q_h^2, b_{21}) &= \hat{q}_h C(\hat{q}_h, b_{21}) + q_h C(\hat{q}_h, b_{21}) \\ &= 2\hat{q}_h C(\hat{q}_h, b_{21}) \end{aligned} \tag{61}$$

Apply equation (32) to $C(\hat{q}_h, b_{21})$ and obtain

$$C(\hat{q}_h^2, b_{21}) = 2\hat{q}_h\{C(c_{20}, b_{21}) + \hat{P}_c C(c_{22}, b_{21}) + \hat{G}C(c_{23}, b_{21}) + \hat{P}V(b_{21})\}.$$

Substitute $V(\hat{q}_h^2)$ and $C(q_h^2, b_{21})$ into equation (58) to obtain

$$V(CS) = V^2(\hat{q}_h)/2b_{21}^2 + \hat{q}_h^2 V(\hat{q}_h)/b_{21}^2 - \hat{q}_h^3\{C(c_{20}, b_{21}) + \hat{P}_c C(c_{22}, b_{21}) + \hat{G}C(c_{23}, b_{21}) + \hat{P}V(b_{21})\}/b_{21}^3 + \hat{q}_h^4 V(b_{21})/4b_{21}^4. \quad (62)$$

When the appropriate numbers are substituted into equation (62), the variance is calculated to equal 36.3124. Recall that the estimate for the consumer surplus area is equal to 18.2538. A standard error of that surplus forecast is equal to 6.0259. The forecast standard error is certainly a more useful reliability measure than the standard errors of the parameters used to calculate that forecast.

Study II

Forecasts of changes in variables and variances

In Study II, an alternate government program is proposed. The wheat support price is raised from \$1.32 to \$1.42 per bushel. So P_{s5} = \$1.42 and dP_s = \$0.10. Because only lagged and time variables are used to estimate the exogenous variable forecasts, they will not be affected. The support price increase will change each of the endogenous variable though. The farm price and government inventory forecasts are directly affected by P_s. The other four variables will change when the farm price difference, $d\hat{P}$, and wheat consumption difference, $d\hat{q}_h$, are considered.

The equations from the Chapter II section on differences of forecasts were used to calculate the forecasts and variances given in Table 4. Since differences are used, this table is interpreted differently than Table 3. Wheat exports, q_E, are forecasted to decrease by 1.8 million bushels if the support price is raised 10 cents. Note that one standard error of that forecast is 1.6 million bushels! The new 1965 export forecast would be 691.6 - 1.8 = 689.8 million bushels.

For an example, the calculations for the change in the wheat price and per capita consumption are given. The b vectors and the D(b) matrices used here are the same as those used in the previous section. Equation (39) is applied to make the forecasts.

$$d\hat{P} = dX_f'b$$

where

$$dX_f' = \begin{bmatrix} 0 & 0.10 & 0 \end{bmatrix} \text{ (a known vector)}$$

$$d\hat{P} = 0.092.$$

The price of wheat is forecasted to increase \$0.092 if the wheat support price is raised \$0.10. The new $\hat{P}$ forecast, $\hat{P}'$, is then \$1.36 + \$0.092 = \$1.45.

Equation (40) is used to calculate the variance of the estimated change.

$$V(d\hat{P}) = dX_f'D(b)dX_f$$

$$= 2.08x10^{-5}.$$

Table 4. Study II: Changes in 1965 forecasts and variances of the estimated changes

Endogenous variables	1965 forecast	Forecast change when $dP_s=0.10$	New 1965 forecast	Forecast variance: Using equation (40)	Forecast variance: Using equation (41)	Ratio of equation (41) to equation (40)
P	1.36	0.092	1.450	2.08×10^{-5} (4.56×10^{-3})[a]	-	-
q_h	2.59	-0.017	2.573	3.36×10^{-5} (5.79×10^{-3})	3.43×10^{-5} (5.86×10^{-3})	1.021
q_f	149.109	-13.664	135.445	11.917 (3.452)	12.408 (3.522)	1.041
C_g	749.23	4.450	753.680	16.687 (4.085)	-	-
C_c	115.63	-6.063	109.567	4.947 (2.224)	5.091 (2.225)	1.029
q_E	691.59	-1.787	689.803	1.932 (1.389)	2.552 (1.597)	1.321

[a]Standard errors are given in parentheses.

The forecast for the change in wheat consumption is given as

$$d\hat{q}_h = dx_f'b$$

where

$$dx_f' = \begin{bmatrix} 0 & 0.092 & 0 & 0 \end{bmatrix}$$

$$d\hat{q}_h = -0.017.$$

Per capita use of wheat for food is forecasted to decrease by 0.017 bushel per annum. The new $\hat{q}_h$ forecast, $\hat{q}_h'$, is 2.59 - 0.017 = 2.57 bushels.

Because $d\hat{P}$ is used to make this forecast, equation (41) must be used to calculate the variance

$$V(d\hat{q}_h) = dx_f'D(b)dx_f + b'D(x_f)b + tr\{D(b)D(x_f)\}$$

where

$D(x_f)$ = 4x4 matrix with 2.08×10^{-5} in the 2nd row, 2nd column position and zeroes elsewhere.

By breaking the variance calculation into its components, it is possible to see the added variability when the forecast is stochastic.

a. $x_f'D(b)x_f = 3.36 \times 10^{-5}$

b. $b'D(x_f)b + tr\{D(b)D(x_f)\} = \underline{7.00 \times 10^{-7}}$

$$3.43 \times 10^{-5} = V(d\hat{q}_h).$$

The last column in Table 4 provides a ratio of the stochastic variance calculation divided by the traditional forecast variance calculation ($3.43 \times 10^{-5}/3.36 \times 10^{-5} = 1.021$). Each of the values in the column can be compared to each other to see how much larger the actual variance is when the reliability of the conditionals is accounted for.

Consumer surplus and its variance

The same measure of consumer surplus that was found in Study I is calculated as a mode of comparison. Using equation (57) with a forecasted value of 2.57 bushesl for $\hat{q}_h'$, consumer surplus equals

$$CS' = -\hat{q}_h'^2/2b_{21} \quad (63)$$

$$= 18.0603.$$

Equation (60) will be used again to calculate the variance of this consumer surplus forecast. But first, $V(\hat{P})$ and $V(\hat{q}_h)$ for the new 1965 forecasts of $\hat{P}$ and $\hat{q}_h$ ($V(d\hat{P})$ and $V(d\hat{q}_h)$ will not suffice. The same procedure from the previous section is used to calculate these. When $\hat{P}'=1.45$

$$V(\hat{P}) = X_f'D(b)X_f + s^2$$

where

$$X_f = \begin{bmatrix} 1 & 1.42 & 0 \end{bmatrix}; \text{ and}$$

$$V(\hat{P}') = 0.0197.$$

When $\hat{q}_h' = 2.57$

$$V(\hat{q}_h') = x_f'D(b)x_f + s^2 + b'D(x_f)b + tr\{D(b)D(x_f)\}$$

where

$$x_f = \begin{bmatrix} 1 & 1.45 & 111.59 & .5271 \end{bmatrix}$$

and 0.0197 is replaced by 0.0198 in $D(x_f)$ (expression (57)).

$$V(\hat{q}_h) = 0.0102$$

These new variance calculations are substituted into equation (60) to obtain $V(\hat{CS}) = 35.5398$. One standard deviation is equal to 5.9615.

Figure 1 summarizes the consumer welfare calculation from Studies I and II.

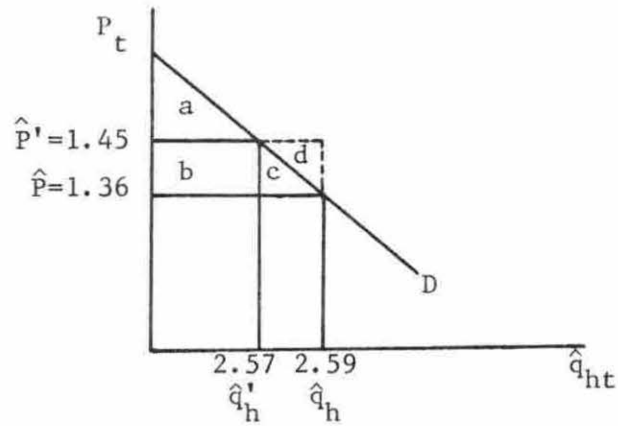

Figure 1. The change in consumer surplus when the wheat support price is raised \$0.10

D is the price dependent demand equation (58). When the support price is \$1.32, $\hat{P}=1.36$ and CS=18.343. This corresponds to areas a+b+c in Fig. 1. When the support price is \$1.42, $\hat{P}'=\$1.45$ and CS'=\$18.047. This corresponds to only area a in Fig. 1. Notice that when the wheat support price is raised, the forecasted price of wheat will increase. This causes estimated consumer surplus to decrease by \$0.2962.

Laspeyre Variation and its variance

Now that $d\hat{P}$ and $d\hat{q}_h$ have been calculated, alternate measures of consumer surplus besides the area under the demand curve can be computed. The Laspeyre Variation is found easily using equation (43). When dP=0.092 and $\hat{q}_h=2.59$ (the quantity demanded before the support price increase):

$$L\hat{V} = \hat{q}_h d\hat{P} \qquad (64)$$

$$= .2375$$

Apply equation (45) to calculate the variance of the forecasted Laspeyre area.

$$V(LV) = \hat{q}_h^2 V(d\hat{P}) + d\hat{P}^2 V(\hat{q}_h) + 2\hat{q}_h d\hat{P} C(\hat{q}_h, d\hat{P}) + V(\hat{q}_h)V(d\hat{P}) + C^2(\hat{q}_h, d\hat{P}) \tag{65}$$

To calculate V(LV), it is first necessary to find $C(\hat{q}_h, d\hat{P})$. The simple recursive model in the Regression Theory section of Chapter II can be applied directly. Set $Y_{1f} = \hat{P}, Y_{2f} = \hat{q}_h$ and $dX_{1f} = dP_S$. Follow the application of equations (33) through (37) to obtain

$$C(\hat{q}_h, d\hat{P}) = b_{21} dP_S \{C(c_{10}, c_{11}) + P_S V(c_{11}) + K\hat{P}_0 C(c_{12}, c_{11})\}. \tag{66}$$

Equation (66) is then substituted into (65) to estimate the exact variance of the Laspeyre Variation.

$$V(\hat{L}V) = d^2\hat{P} V(\hat{q}_h) + \hat{q}_h^2 V(d\hat{P}) + 2\hat{q}_h d\hat{P} b_{21} dP_S \{C(c_{10}, c_{11}) + P_S V(c_{11}) + K\hat{P}_0 C(c_{12}, c_{11})\} + V(\hat{q}_h)V(d\hat{P}) + b_{21}^2 d^2 P_S \{C(c_{10}, c_{11}) + P_S V(c_{11}) + K\hat{P}_0 C(c_{12}, c_{11})\}^2. \tag{67}$$

With dP=0.092, $\hat{q}_h$=2.57, all the appropriate substitutions are made to yield an estimate for the variance of 2.786×10^{-4}. One standard error is equal to 0.0167.

Consumer gain and its variance

Consumer gain is another measure which is used to estimate consumer surplus. Use equation (47) and the estimate for LV to calculate an estimate for CG. When $d\hat{q}_h$=-0.017 and $d\hat{P}$=0.092,

$$C\hat{G} = \hat{q}_h d\hat{P} + dq_h dP/2$$
$$= L\hat{V} + (-.017)(.092)/2 \quad (68)$$
$$= .2367$$

Note that the primary component of the consumer gain area is equal to the Laspeyre Variation. This will slightly simplify the derivation of the Consumer Gain variance formula. From equation (48)

$$V(CG) = V(LV) + V(d\hat{q}_h d\hat{P})/4 + C(\hat{q}_h d\hat{P}, dq_h dP). \quad (69)$$

Some new problems are encountered when evaluating the terms in this equation. The data do not fit the assumptions of the statistical procedures which have been used throughout the thesis. The variance and covariance formulas presented (equations (8) through (10)) apply to products of joint normal variables. But the $\hat{q}_h$ and $d\hat{q}_h$ estimates are not normally distributed because they are functions of the stochastic variables $\hat{P}$ and $d\hat{P}$. Neither are $\hat{q}_h$ and $d\hat{q}_h$ independent of $\hat{P}$ and $d\hat{P}$. General formulas for jointly distributed random variables presented by Bohrnstedt and Goldberger require knowledge of third moments. So, some specification error is admittedly present because the formulas for jointly normal variables are being applied to variables that are not jointly normal.

In equation (69), V(LV) is equation (67). Equation (8) is used to compute $V(dq_h dP)$.

$$V(d\hat{q}_h d\hat{P}) = d^2\hat{q}_h V(d\hat{P}) + d^2\hat{P}V(d\hat{q}_h) + 2d\hat{q}_h d\hat{P}C(d\hat{q}_h, d\hat{P}) + V(d\hat{q}_h)V(d\hat{P}) + C^2(d\hat{q}_h, d\hat{P}) \quad (70)$$

To find the formula for equation (70), $C(dq_h, dP)$ must first be computed. In this situation, $d\hat{q}_h$ depends upon $d\hat{P}$. In the notation of equations (20) and (21), this means that dy_{2f} depends upon dy_{1f}. But, in fact, dy_{2f} does not depend upon dy_{1f}. So, the equations can be reformulated to allow for this interdependence. The result yields a complicated expression because u_1 affects b_1 and also b_2, which makes $D(b_1, b_2)$ more complex. It is easier to use the structural models for P and q_h along with equation (25) to compute

$$
\begin{aligned}
C(d\hat{q}_h, d\hat{P}) &= C(b_{21} d\hat{P}, d\hat{P}) \\
&= b_{21} V(d\hat{P}).
\end{aligned}
\tag{71}
$$

Equation (71) is substituted into (70) to yield $V(d\hat{q}_h, d\hat{P})$.

The next term in equation (69) which must be considered separately is $C(q_h dP, dq_h dP)$. Use equation (10) with y=v to evaluate this expression.

$$
\begin{aligned}
C(\hat{q}_h d\hat{P}, d\hat{q}_h d\hat{P}) = {} & \hat{q}_h d\hat{q}_h V(d\hat{P}) + \hat{q}_h d\hat{P} C(d\hat{P}, d\hat{q}_h) + d\hat{P} d\hat{q}_h C(\hat{q}_h, d\hat{P}) + \\
& d^2\hat{P} C(\hat{q}_h, d\hat{q}_h) + C(\hat{q}_h, d\hat{q}_h) V(d\hat{P}) + \\
& C(\hat{q}_h, d\hat{P}) C(d\hat{P}, d\hat{q}_h)
\end{aligned}
\tag{72}
$$

Equation (66) provides the expression for $C(\hat{q}_h, d\hat{P})$. To obtain $C(\hat{q}_h, d\hat{q}_h)$ use equation (21) with $d\hat{q}_h = dy_{2f} = dx'_{2f} b_2$ and $q_h = y_{2f} = x'_{2f} b_2$. Then

$$
\begin{aligned}
C(\hat{q}_h, d\hat{q}_h) = {} & dx_{2f} D(b_2) x_{2f} + b_2 D(dx_{2f}, x_{2f}) b_2 + \\
& tr\{D(b_2) D(dx_{2f}, x_{2f})\}.
\end{aligned}
\tag{73}
$$

The only nonzero element in dx_{2f} is $d\hat{P}$. Similarly, $C(d\hat{P}, \hat{P})$ is the only nonzero element in $D(dx_{2f}, x_{2f})$.

Equation (20) can be applied to compute $C(d\hat{P},\hat{P})$. This is because dP_S, the only nonzero element of dx_{1f}, is a known constant. So, $D(dx_{1f},x_{1f})=0$. Consequently,

$$C(d\hat{P},\hat{P}) = dP_S\{C(c_{10},c_{11}) + P_SV(c_{11}) + K\hat{P}_0C(c_{11},c_{12})\}. \tag{74}$$

Equation (73) can then be expressed as

$$C(\hat{q}_h,d\hat{q}_h) = d\hat{P}\{C(c_{20},b_{21}) + \hat{P}_cC(c_{22},b_{21}) + \hat{G}C(c_{23},b_{21}) + \hat{P}V(b_{21})\} + \{b_{21}^2 + V(b_{21})\}\{C(c_{10},c_{11}) + P_SV(c_{11}) + K\hat{P}_0C(c_{11},c_{12})\}dP_S. \tag{75}$$

Equations (66), (70), and (75) are appropriately substituted into (72) to obtain the correct equation for $C(\hat{q}_hd\hat{P},d\hat{q}_hd\hat{P})$.

$$C(q_hdP,dq_hdP) = \hat{q}_hd\hat{q}_hV(d\hat{P}) + \hat{q}_hd\hat{P}b_{21}V(d\hat{P}) + \begin{bmatrix} C(c_{10},c_{11})+P_SV(c_{11})+ \\ K\hat{P}_0C(c_{11},c_{12}) \end{bmatrix} \times \begin{bmatrix} dPdq_hb_{21}dP_S+\{b_{21}^2+V(b_{21})\} \\ \{d^2PdP_S+V(d\hat{P})dP_S\}+ \\ \{b_{21}^2d\hat{P}_SV(d\hat{P})\} \end{bmatrix} + \begin{bmatrix} C(c_{20},b_{21}+\hat{P}_cC(c_{22},b_{21}) \\ +\hat{G}C(c_{23},b_{21})+\hat{P}V(b_{21}) \end{bmatrix} \times \begin{bmatrix} d^3P+dPV(dP) \end{bmatrix} \tag{76}$$

To complete the derivation of V(CG), equations (67), (70), and (76) must be substituted into equation (69). When all the appropriate formula and variable substitutions are made, an estimate for the variance was calculated to equal $2.770\text{x}10^{-4}$. One standard error is equal to 0.0166.

This variance for the Consumer Gain estimate is very close to the variance of the Laspeyre Variation estimate. The actual estimates for LV and CG are also very close (see Table 5).

Fig. 1 can be used again to compare the calculated consumer surplus areas. Recall that D is defined as the Marshallian (or uncompensated) demand curve. If the assumption is made that the consumer's gain is offset by a continuous application of the compensating variation, then the correct area for CG would be measured under a compensated demand curve. In defining CG though, Winch (1965) notes that if the assumption is made that no compensating variation is actually made, then the consumer's gain is measured by the area under the uncompensated demand curve. Using this assumption, the measured area of consumer gain, 0.2367, therefore corresponds to b+c in Fig. 1. This estimate for CG should equal the difference between the two estimates of the areas under the demand curve before and after the price increase (18.2558-18.0603=0.1954). Since this difference is only one percent of the actual estimates, it is possible to attribute the discrepancy between the two area estimates, 0.2367 and 0.1954, to rounding error.

Table 5 can also be used to compare the variance estimates of the consumer surplus areas. These variance measures can be used to judge "the reliability" of the area estimates. Note the small magnitude of the V(LV) and V(CG) estimates. One would incorrectly assume that the slightly smaller variance calculation for Consumer Gain resulted because the CG area is less than the LV area. This is because the variance (i.e. the reliability) of any forecasted change in consumer surplus is not known and cannot be estimated until it is actually calculated. The consumer surplus variance

Table 5. Consumer surplus measures and then variance[a]

Consumer surplus measure	Study I		Study II (when dP_s=0.10)	
	Estimate	Variance	Estimate	Variance
CS	18.2558	36.3124 (6.0259)	18.0603	35.5398 (5.9615)
LV	-	-	0.2375	$2.786x10^{-4}$ (0.0167)
CG	-	-	0.2367	$2.770x10^{-4}$ (0.0166)

[a]Standard errors are given in parentheses.

formulas given in this section show that it is possible to estimate the exact variance of forecasted consumer welfare changes from a predicted policy action.

Study III

The objective of Study III is to empirically show how much less reliable the variable forecasts become as the target date moves further into the future. Recall that in Study I, 1965 forecasts were made under the assumption that the same price support program was maintained. The same assumption is made in this study, i.e. $P_{s4}=P_{s5}=P_{s6}$=\$1.32 per bushel. The same recursive procedure from Study I is also applied to calculate the 1966 forecast.

All of the forecasts made in this study are stochastic. That is, 1965 forecast values are used to make the 1966 predictions. The functional form for the exogenous variables is given by

$\hat{X}_6 = f(\hat{X}_5, T_6)$

where $\hat{X}_5$ is the estimate obtained from Study I. Each forecast of the endogenous variables is also stochastic. The exception remains for the wheat price forecast though. $\hat{P}_6$ will be an unconditional forecast since the support price is assumed known, and $K_5=K_6=0$, so $K_6\hat{P}_{o6}=0$.

All of the calculated results from Study III are presented in Table 6. For ease of comparison, the columns of interest from Table 3 are also included. It is possible to see how the 1966 forecast variance of some variables greatly increased from their 1965 variance values.

In this study, the example forecast and variance calculations for the wheat price and per capita consumption will directly follow those given in Study I. The exact same procedure is used to derive the 1966 price forecast. $\hat{P}_6$ will equal the $\hat{P}_5$ forecast because the X_f vector does not change. This implies that the variance for $\hat{P}_6$, $V(\hat{P}_6)$, is also the same (see Table 6).

The calculations for the 1966 consumption forecast, $\hat{q}_{h6}$, and forecast variance, $V(\hat{q}_{h6})$, are different from those in Study I because the stochastic independent variables will change. Following the same procedure, the necessary information is

1966 wheat price forecast	$\hat{P}_6$	= 1.36
The variance of the price forecast	$V(\hat{P}_6)$	= 0.0198
1966 consumer price index forecast	$\hat{P}_{c6}$	= 114.37
The variance of the price index forecast	$V(\hat{P}_{c6})$	= 12.24
1966 disposable income forecast	$\hat{I}_6$	= 2401.94
The variance of the income forecast	$V(\hat{I}_6)$	= 4687.44

Table 6. Study III: Forecasts and variances

Exogenous variables	1965 forecast	1965 forecast variance	1966 forecast	Forecast variance: Using equation (5)	Forecast variance: Using equation (13)	Ratio of equation (13) to equation (5)
P_o	109.15	507.54	115.69	507.25 (22.52)[a]	794.16 (28.18)	1.566
P_c	111.29	7.08	114.37	7.15 (2.67)	12.24 (3.50)	1.712
L	169.00	71.07	170.13	71.63 (8.46)	110.25 (10.50)	1.539
O	1334.65	24,270.17	1333.40	24,538.58 (156.65)	29,158.05 (170.76)	1.188
I	2334.86	2,777.00	2401.94	2,816.25 (53.07)	4,684.44 (68.74)	1.663
Endogenous variables						
P	1.36	0.0198	1.36	0.0198 (0.14)	-	-
q_h	2.59	0.0162	2.56	0.0144 (0.12)	0.0174 (0.13)	1.208

[a]Standard errors are given in parentheses.

Table 6. (Continued)

Endogenous variables	1965 forecast	1965 forecast variance	1966 forecast	Forecast variance: Using equation (5)	Forecast variance: Using equation (13)	Ratio of equation (13) to equation (5)
q_f	149.11	5,272.94	162.19	3,046.72 (55.19)	6,416.35 (80.10)	2.106
C_g	749.23	24,069.4	783.55	23,543.12 (153.44)	39,205.46 (198.00)	1.665
C_c	115.63	3,704.77	120.91	3,485.83 (59.04)	4,249.84 (65.19)	1.219
q_E	691.59	10,725.42	674.78	10,229.58 (101.14)	15,979.11 (126.41)	1.562

Income variable transformation (using (54)) $G(\hat{I}_6) = 0.4961$

The variance of the transformation $V(\hat{G}_6) = 9.45 \times 10^{-4}$

The estimated MSE from equation (7) $s^2 = 0.0107$

The 4x4 dispersion matrix of the estimated coefficients in equation 7 (Table 2)

The 4x4 dispersion matrix of the forecasted independent variables

$$D(x_f) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & 0.0198 & 0 & 0 \\ & & 12.24 & 0 \\ \text{symm.} & & & 9.45 \times 10^{-4} \end{bmatrix}$$

The independent variable forecast variances used in $D(x_f)$ are those calculated using equation (13). Since the forecast for $\hat{q}_{h6}$ is conditional upon stochastic forecasts, all variability which might be present in $\hat{q}_{h6}$ must be accounted for.

The 1966 wheat consumption forecast is

$$\hat{q}_{h6} = x_f' b$$

where

$$x_f' = \begin{bmatrix} 1 & 1.45 & 114.37 & 0.4961 \end{bmatrix}$$

$$b = \begin{bmatrix} 1.6656 & -0.1834 & 0.0040 & 1.3857 \end{bmatrix}$$

$$\hat{q}_{h6} = 2.56 \text{ bushels per capita.}$$

The exact variance of this forecast is given by

$$V(\hat{q}_{h6}) = x_f' D(b) x_f + s^2 + b' D(x_f) b + \text{tr}\{D(b) D(x_f)\}.$$

Breaking the variance calculation into its components yields:

a. $x'_f D(b) x_f + s^2 \quad = 0.0144$

b. $b' D(x_f) b \quad = 2.674 \times 10^{-3}$

c. $tr\{D(b) D(x_f)\} \quad = \underline{3.257 \times 10^{-4}}$

$\quad 0.0174 \quad = V(\hat{q}_{h6}).$

The conventional variance measure, line a, is equal to 0.0144. Note that this value is less than the exact variance for the 1965 forecast, $V(\hat{q}_{h5})$=0.0162. This is the case for all of the other endogenous variables also. Notice, however, when the variance of the conditionals is included in the calculations, all of the variance measures increase. Comparison of the last columns in Tables 3 and 6 indicate that a lot more forecast variability is present when the independent variables are themselves forecasts. The magnitude of the values in the columns also reiterates the point that a lot of variability is left unexplained when only the traditional forecast variance formula is used.

CHAPTER IV. CONCLUSION

Any forecast economist who publishes a market study would certainly hope that their work is a worthwhile effort and that it will be used by others with interest in the field. It is unfortunate that those who do use the studies are not fully informed of the reliability of the forecasts which are made. Many economists conclude their forecast evaluations with the goodness of fit of the models which were used to make the forecasts. A confidence statement about the forecast itself is seldom reported.

When a more precise reliability measure is calculated, it is usually in the form of a tolerance interval. Computation of the interval does require calculating the forecast variance, which is done incorrectly when the forecast is conditional. Forecast variance formulas found in econometrics texts fail to consider the additional variability present when a forecast is used to make another forecast. As a result, the reliability of the stochastic forecast is overstated. The variance calculations done for forecasts of variables in the wheat sector verify this statement.

The textbook formula was applied to calculate the variance of some 1965 forecasts and also some 1966 forecasts of variables. In some cases, the 1966 forecast variances were smaller than those for 1965. It is apparent that this formula fails to measure any deterioration in the quality of forecasts as the target date moves futher in the future.

When the forecast variances were recalculated to account for the stochastic independent variables, then every variance increased.

For one variable, the use of wheat for feed, q_f, the 1966 forecast variance doubled when conditional variances were included in the calcuations. One would incorrectly conclude that the variance increased so much because the estimated model only explained 78 percent of the variation in q_f. In equations with lower R^2's, the increase in variance was minimal. The increase in variance of forecasted changes in variables was also very small when the stochastic variables were included in the calculations. The point is that it is not possible to judge exactly how much less reliable a stochastic forecast is, until the variance of that forecast is correctly computed.

Another finding from this study was that it is possible to derive variance formulas for some commonly used measures of consumer surplus. It is hoped that these original formulas will be used throughout studies of consumer welfare. An estimate of the exact variance of a forecasted loss in consumer surplus is certainly a more informative reliability measure than the usual subjective confidence estimate which the author choses to report.

The variance formula for a stochastic forecast is a very useful measure which needs to be applied in a lot more econometric studies. Likewise, the variances of the estimated consumer surplus measures are also useful reliability measures which should be used in every study of predicted policy consequences.

REFERENCES

Bohrnstedt, George W. and Goldberger, Arthur S. "On the Exact Covariance of Products of Random Variables." Journal of the American Statistical Association 64 (1969):1439-1442.

Brown, T. M. "Standard Errors of a Complete Econometric Model." Econometrica 22 (1954):178-192.

Chambers, Robert G. and Just, Richard E. "Effects of Exchange Rate Changes on U.S. Agriculture: A Dynamic Analysis." American Journal of Agricultural Economics 63 (1981):32-46.

Cory, Dennis C., Gum, Russell L., Martin, William E., and Leigh, Marie. "Use of Paasche and Laspeyres Variations to Estimate Consumer Welfare Change." Agricultural Economic Research 33 (No. 2) (1981):1-6.

Currie, John M., Murphy, John A., and Schmitz, Andrew. "The Concept of Economic Surplus and its Use in Economic Analysis." Economic Journal (1971):

Dhrymes, Phoebus J., Howrey, E. Philip, Hymans, Saul H., Kmenta, Jan, Leamer, Edward E., Quandt, Richard E., Ramsey, James B., Shapiro, Harold T., and Zarnowitz, Victor. "Criteria for Evaluation of Econometric Models." Annals of Economic and Social Measurement 1/3 (1972):291-323.

Feldstein, Martin S. "The Error of Forecast in Econometric Models when the Forecast-Period Exogenous Variables are Stochastic." Econometrica 39 (1971):55-60.

Hooper, J. W. and Zellner, A. "The Error of Forecast for Multivariate Regression Models." Econometrica 29 (1961):544-555.

Johnston, J. Econometric Methods (2nd ed.). New York: McGraw-Hill, 1972.

Ladd, George W. "Variances of Products of Forecasts." Unpublished paper, Department of Economics, Iowa State University, undated.

Mo. William Y. "An Economic Analysis of the Dynamics of the United States Wheat Sector." Washington, D.C.: Government Printing Office, Technical Bulletin No. 1395, 1968.

Parker, Russell C. and Connor, John M. "Estimates of Consumer Loss Due to Monopoly in the U.S. Food-Manufacturing Industries." American Journal of Agricultural Economics (1979):626-638.

Pindyck, Robert S. and Rubinfeld, Daniel L. Econometric Models and Economic Forecasts. New York: McGraw Hill, 1981.

Ramsey, James B. Economic Forecasting - Models or Markets? London: The Institute of Economic Affairs, 1977.

Rao, C. Radhakrishna. Linear Statistical Inference and its Applications (2nd ed.). New York: John Wiley, 1973.

Snedecor, George W. and Cochran, William G. Statistical Methods (7th ed.). Ames, Iowa: The Iowa State University Press, 1980.

Williams, W. H. and Goodman, L. M. "A Simple Method for the Construction of Empirical Confidence Limits for Economic Forecasts." Journal of the American Statistical Association 66 (1971):752-754.

Winch, David M. "Consumer's Surplus and the Compensation Principle." American Economic Review 55 (1965):395-423.

ACKNOWLEDGEMENTS

The main person that I have got to thank for my being where I am today is Dr. John VanSickle. It was his convincing which persuaded me to go to graduate school in the first place. I often claimed it was his fault that I ended up so far from home, but deep down, I am truly grateful. I know now, as he tried to tell me years ago, that Dr. George Ladd is a good major professor. When the deadlines came, Dr. Ladd was there to help me. It is from his inspiration and guidance that makes me feel like a semi-accomplished economist. He has taught me two valuable lessons - to be skeptical about givens and that we can never know everything about anything.

I am forever indebted to by social support group of friends here in Ames. I can't count the number of times they have helped me get my mind off my troubles and made me laugh. I have really enjoyed every single Friday with the Yacht Club. Thank you Marty, Jim, Dave, Jane, Mark, Trish, Everett, Julie, Dave, Mary, George, Kim, Mark W., Kesavan, Dan, Elwin, Darold, Debbie, Gary, Ken, and Katie. Finally, thank you Diane and Damona.

My thanks go to Damona for being the original Physics-Econ. go-between and also for her model perserverance. I know you will go far. I am also thankful to my friend, Diane, who's been with me from day one in graduate school. I know our sharing will continue for a long time.

I want to thank my family so much for their proud support which has helped me many times to just keep going and finish my degree. I know it

was a strain to all of us for me to come to Iowa to go to school, but we adjusted. Thanks Dad for your letters, and Mom, for your love. To Tina and Paul - I only hope you will let me help both of you some time the way you helped me this past year. Being on your own is not that difficult when you have well-wishers behind you.

This final space I have reserved for Rick. Since I know you quietly objected to my leaving Gainesville in the first place, I have to thank you for your patient understanding. I know our belief in each other will help each of us through the many trials that face us in the future. Thank you for your love.